— Exercises —

1. **Interpretation of partial derivatives.** Let $f : \mathbb{R}^n \to \mathbb{R}$. Check that $D_j f(x_1, \ldots, x_n)$ is the derivative at x_j of the function $t \mapsto f(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)$.

2. **Linear forms.** Let $L : \mathbb{R}^n \to \mathbb{R}$ be a linear map (such an L is called a linear form). Write $L(x_1, \ldots, x_n)$ more explicitly. Why can we say that a linear form $L : \mathbb{R} \to \mathbb{R}$ is "just a number"?

3. **The derivative of a linear map is the map itself.** Let $L : \mathbb{R}^n \to \mathbb{R}$ be a linear form. Show that L is differentiable, and that $\forall c \in \mathbb{R}^n, dL(c) = L$.

4. **A computation of partial derivatives.** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a differentiable function. We define a function f on $\mathbb{R}^* \times \mathbb{R}$ by $f(x, y) = \varphi(\frac{y}{x})$. Show that f satisfies the relation

$$xD_1 f(x, y) + yD_2 f(x, y) = 0$$

at any point $(x, y) \in \mathbb{R}^* \times \mathbb{R}$.

— Problems —

5. **Directional derivatives vs continuity.** Let f be the function defined on $\mathbb{R}^2$ by

$$f(x, y) = \begin{cases} \frac{y^2}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f admits directional derivatives at $(0, 0)$ in any direction. Is f continuous at $(0, 0)$?

6. **Partial and directional derivatives vs continuity.**

   (a) (Partial derivatives, but not continuity.) Consider the function

$$\begin{aligned} f : \ \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \end{aligned}$$

   Show that at the point $(0, 0)$, the partial derivatives exist but f is not continuous.

   (b) Using this f, check that in general, existence of all directional derivatives at some point c is a stronger requirement than existence of all partial derivatives at c.

7. **A function which is $\mathcal{C}^1$ but not $\mathcal{C}^2$.** Let f be the function defined on $\mathbb{R}^2$ by

$$f(x, y) = \begin{cases} \frac{xy^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that f is $\mathcal{C}^1$. Compute $D_1 D_1 f$ and show that it is not continuous.

8. **Differentiability vs. continuous differentiability.** Let f be the function defined on $\mathbb{R}^2$ by

$$f(x, y) = \begin{cases} xy \sin(\frac{1}{\sqrt{x^2+y^2}}) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that f is differentiable at $(0, 0)$ but not $\mathcal{C}^1$.